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LETTER TO THE EDITOR

On the scaling properties of various invasion models

L de Arcangelis† and H J Herrmann†‡

† SPHT, CEN Saclay, F91191 Gif-sur-Yvette, France
‡ HLRZ, KFA Jülich, Postfach 1913, 5170 Jülich, Federal Republic of Germany

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Abstract. We investigate the multiscaling behaviour of usual and directed invasion percolation, with and without a power-law decay with the distance from an initial seed in the distribution of random numbers. We find universal multiscaling behaviour only in the presence of a power-law gradient in space.

Invasion percolation [1] is a geometrical growth model that has been used to describe the penetration of a fluid into a porous medium. The clusters of penetrating fluid created by invasion percolation evolve automatically into fractals indistinguishable from the incipient infinite clusters at the critical threshold of usual percolation. For this reason invasion percolation constitutes an ideal example of self-organized criticality.

One of the most interesting questions about growth models is the scaling behaviour of their growth zone. In many cases, like the Eden model [2] or epidemics [3], a new set of growth exponents is found. Even more complicated seems to be the situation in nucleation [4] or diffusion limited aggregation (DLA) [5] where recently a new type of scaling, called *multiscaling*, has been proposed. In this case the effective fractal dimension continuously varies radially within the growth zone.

The growth of invasion percolation clusters occurs as a sequence of jumps [6], and it is therefore likely that this behaviour reflects in the scaling properties of the growth zone. It is the purpose of this letter to investigate the radial dependence of the fractal dimension for various invasion percolation models. Besides the standard version of the model without trapping [1] we also consider directed invasion percolation and invasion percolation with a spatially graded distribution of random numbers.

Invasion percolation is defined by placing on each site of a regular lattice a random number $z_i \in (0, 1)$ uncorrelated from site to site. Then one chooses in the centre of the lattice a site, the seed of the growth, and *occupies* this site. Finally one grows the cluster by applying over and over again the following rule: one chooses among all the sites that are nearest neighbours of occupied sites the one which has the smallest value of z, and occupies it. On a finite lattice the growth is stopped when a site on the boundary is occupied.

The ensemble of occupied sites produced with this algorithm is necessarily a single cluster, i.e. each site is connected to any other site by a path of nearest-neighbouring occupied sites. It has been convincingly argued and shown numerically, but not yet rigorously proven, that the fractal dimension d_f of this cluster equals the fractal

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dimension of the spanning cluster at p_c in standard percolation, namely $d_f = 91/48$ in two dimensions.

Besides the usual invasion percolation without trapping, described above, we also study here the directed case. This implies that each bond on the lattice allows occupation in one direction only and not in the other. In our case we consider a square lattice with all horizontal bonds directed to the right and all vertical bonds directed downwards. In the description of a fluid penetrating a porous medium this means that the channels between pores have valves that allow the fluid to flow in one direction only.

The algorithmic implementation of directed invasion percolation is straightforward: the site chosen to be occupied must not just be a nearest neighbour of an already occupied one, but the occupied neighbour must also be either up or to the left in the case of a square lattice.

The random numbers z assigned at the beginning to the sites of the lattice are usually uniformly distributed. We study here also the effects of radially graded random numbers. The purpose is to investigate either the effect of a pressure gradient, or a radial variation in the structure of the porous medium. In order to cover a wide range of different distributions we consider a power-law decay in the value of the random numbers as function of the distance from the seed, that is

$$z(r) = z_0 r^{-\alpha} \tag{1}$$

where z_0 is a random number uniformly distributed between zero and one.

The effect of such a gradient depends on the sign of α. If α is negative the clusters tend to be compact like Eden clusters. More interesting is the case of positive α, which we analyse in detail for both usual and directed invasion percolation. The clusters grow in one direction only, but the direction itself depends on the configuration of random numbers. In figure 1 we show one cluster obtained for usual invasion percolation with $\alpha = 1$.

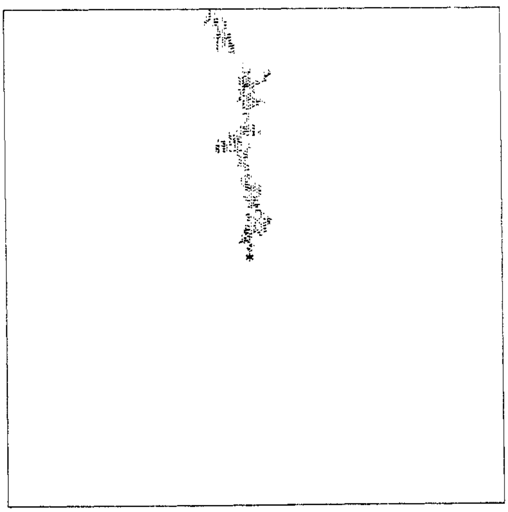

Figure 1. Invasion percolation cluster grown in the presence of a gradient with $\alpha = 1.0$ on a cubic lattice of linear size $L = 400$. The growing seed is placed in the centre of the lattice (*).

We know that usual invasion percolation clusters are fractals, i.e. that the number M of sites in the cluster scales as

$$M \sim R^{d_f} \tag{2}$$

where R is the radius of gyration of the cluster.

Directed clusters, like directed percolation clusters or directed animals, follow an anisotropic scaling law: their mass M growths with their characteristic length $\xi_\parallel$ as

$$M \sim \xi_\parallel^{1/\nu_\parallel} \tag{3a}$$

whereas their width $\xi_\perp$ scales as

$$\xi_\perp \sim \xi_\parallel^{1/\theta} \tag{3b}$$

where θ is often expressed as $\theta = 1 + \nu_\parallel / \nu_\perp$. For directed percolation one has $\theta \simeq 1.58$ and $\nu_\parallel \simeq 0.64$ in two dimensions [7]. We expect our directed invasion percolation clusters to have this behaviour and probably also the clusters obtained in the gradient of (1) will follow for $\alpha \neq 0$ a scaling of this type.

If instead of the total mass M of the cluster one considers the mass within concentric shells around the seed, a more general scaling ansatz, called *multiscaling*, has been proposed [4]. Let r be the radius of the shell and R the radius of gyration of the cluster. For fixed values of the parameter $x \equiv r/R$ the mass $m(r, R)$ in the shell scales as

$$m(r, R) = r^{d_f(x)} F(x) \tag{4}$$

where $F(x)$ is a scaling function. If $d_f(x)$ is a constant the normal scaling (2) is recovered. On the other hand, if $d_f(x)$ is a function of x as it was found for nucleation [4] and DLA [5] one has an infinity of fractal dimensions, namely multiscaling.

Typically multiscaling is due to the fact that at some distance from the seed the cluster has not finished growing and therefore the effective fractal dimension in these unfinished regions has not yet converged to the value expected for d_f in the centre of an infinitely large cluster. Next we investigate this question for the various invasion percolation models introduced.

We have performed numerical simulations on the square lattice for standard and directed invasion percolation. Once an initial seed is placed on the lattice, the clusters are grown following the algorithms discussed above. We analyse both the case of the random numbers z drawn from the uniform distribution, and the case of a radial gradient in the value of z. To this extent we consider the power law in (1) for values of $\alpha = 0.1$ and 1.0. The whole simulation took about 130 hours of CPU time on a Cray XMP.

Once the cluster reaches the boundary of the lattice the growth is stopped and the multiscaling analysis is performed. This analysis of the clusters is also done during the growth process, more precisely each time 500 new sites are added to the cluster, in order to have data for a wide range of values of the radius of gyration R. For each cluster we calculate the radius of gyration R and then the mass $m(r, R)$ in a shell at a distance $r = xR$ from the initial seed. In practice, we choose a sequence of values of x, ranging from 0.1 to about 2.0, and determine for each value of x the number of points whose distance from the seed is between $r = xR$ and $r' = x'R$ where $x' = x + 0.1$.

The log-log plot of the average mass $\langle m(r, R) \rangle$ against the average radius $\langle R \rangle$ for fixed x provides the fractal dimension $d_f(x)$ defined in (4), as seen in figure 2. As x increases from 0.1 to 2.0 one successively explores the regions of the cluster starting

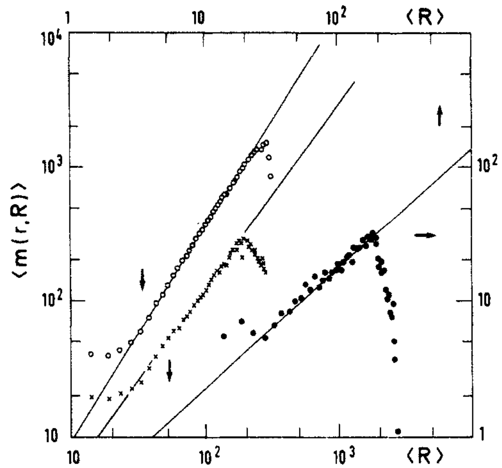

Figure 2. Log-log plot of the average mass in a shell $\langle m(r, R)\rangle$ against the average radius of gyration $\langle R\rangle$ for 1500 configurations of invasion percolation clusters grown with the gradient exponent $\alpha = 0.1$ on a lattice of linear size $L = 760$. The values of the slope $d_f(x)$ are: 1.575 for $x = 1.6$ (○), 1.356 for $x = 1.8$ (×) and 0.890 for $x = 2.0$ (●).

from the ones very close to the seed and whose growth is settled, up to the very external regions belonging to the last growing front. If any multiscaling behaviour is then present in this problem, one expects to find it in the x dependence of $d_f(x)$.

Figure 3 shows the fractal dimension $d_f(x)$ as function of x for standard invasion percolation and in the presence of a radial gradient. For the standard case ($\alpha = 0.0$) the fractal dimension is constant within statistical fluctuations and close to the expected value $d_f \simeq 1.89$.

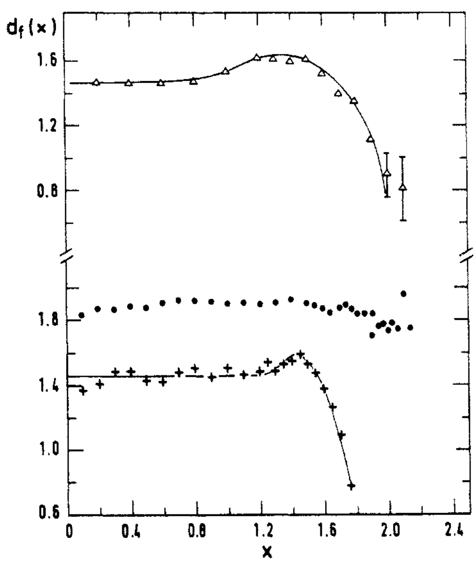

Figure 3. The fractal dimension $d_f(x)$ as function of x for 800 configurations of standard invasion percolation clusters (●); invasion percolation clusters in the presence of a radial gradient with $\alpha = 0.1$ (1500 configurations, △) and $\alpha = 1.0$ (4000 configurations, +). The linear size of the lattice on which the clusters are grown is $L = 760$.

In the case of a radial gradient in the random numbers, the data exhibit instead a richer behaviour, namely multiscaling. The log-log plot of $\langle m(r, R)\rangle$ as function of $\langle R\rangle$ shows a continuously varying slope for various values of x (figure 2). As the system size increases the data are linear over a wider range of R. A deviation from the linear behaviour is observed for small clusters, whereas the sharp bending of the data for large radii is due to the finite size of the lattice on which the clusters are grown. Moreover, the stronger statistical fluctuations for large x (for instance, $x = 2.0$ in figure 2) are due to the small number of clusters giving contributions at distances so far from the initial seed ($r = 2R$).

The curve of fractal dimensions exhibits an initial plateau at a value of $d_f(x)$ close to 1.47 ± 0.04. This exponent seems to be independent within error bars of the strength of the radial gradient, i.e. the value of α chosen, for $\alpha > 0$. It is therefore a new universal exponent found for invasion percolation in the presence of a spatial gradient. The $d_f(x)$ curve has then a maximum at $x \simeq 1.4$ and decays towards zero for larger values of x. The general shape of $d_f(x)$ is similar to the one found for DLA [5] in two dimensions, where the plateau is found at the well known value of the fractal dimension $d_f \simeq 1.7$.

We have performed the same analysis for the directed invasion percolation clusters. Again it is found that in the standard model $d_f(x)$ is about constant within error bars and close to 1.44 ± 0.02 (figure 4). This value is consistent with the expected fractal dimension for two-dimensional directed percolation, where $d_f = 1/\nu_\parallel \simeq 1.56$ [7]. In the presence of a radial gradient in the random numbers, the plateau shifts down to a value 1.29 ± 0.03, a novel exponent for directed invasion percolation, and $d_f(x)$ again seems to decrease for x larger than 1.6. Compared to the case of standard invasion percolation with a gradient though, the data here exhibit a less sharp decay and, due to rather large error bars for large x, we cannot completely exclude the validity of simple scaling, with only one fractal dimension, for the directed invasion percolation cluster with a radial gradient.

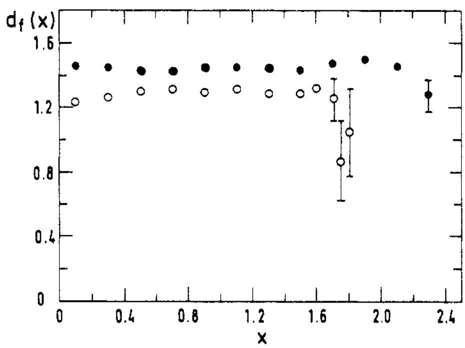

Figure 4. The fractal dimensions $d_f(x)$ as function of x for 500 configurations of standard directed invasion percolation clusters (●); and 7000 clusters in the presence of a radial gradient with $\alpha = 1.0$ (○). The linear size of the lattice is $L = 660$.

In conclusion, multiscaling has been recently found in two rather different problems: in DLA, a far from equilibrium dynamical process; and in spinodal decomposition, a relaxation process towards equilibrium. The common feature between these two problems is the presence of two characteristic lengths, having the same scaling behaviour but differing by a logarithmic factor. For instance, in DLA these two lengths are the radius of gyration of the cluster and the width of the growing interface [2]. One might therefore infer that multiscaling appears typically when dealing with growth processes.

In this letter we have introduced a new class of models for standard and directed invasion percolation, where the growing probabilities are function of the distance from the initial seed. These models could be used to study the flow of a fluid in a porous medium in the presence of a pressure gradient. For the various invasion models we find that simple scaling, with a single fractal dimension, is followed when the distribution of random numbers, and therefore of growing probabilities, is uniform for all perimeter sites.

However, if a radial gradient is introduced in the distribution, the growth of the cluster becomes faster and therefore more unstable and farther away from equilibrium. In this case multiscaling occurs and an infinite number of fractal dimensions is necessary to characterize the geometrical properties of the cluster. The values of the fractal dimensions found for these models are new scaling exponents, universal with respect to strength of the spatial gradient for $\alpha > 0$.

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